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Dependence of α in Peak Norms and Best Peak Norms Approximation

Chengmin Yang

Department of Mathematics, West Virginia University, Institute of Technology, Montgomery, West Virginia 25136
E-mail: chyang@wvit.wvnet.edu

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Let $C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$ and U be an n dimensional subspace of $C[0, 1]$. A peak norm, or α-norm for $0<\alpha\leqslant 1$, α-norm is defined by $\|f\|_\alpha=\frac{1}{\alpha}\sup\{\int_A |f|\,d\mu \mid \mu(A)=\alpha,\ A\subset[0,1]\}$, where μ denotes the Lebesgue measure. We say $p\in U$ is a best α-norm approximant to f from U if $D_\alpha(f)=\|f-p\|_\alpha=\inf\{\|f-u\|_\alpha \mid u\in U\}$. In this paper we shall study $\|f\|_\alpha$, $D_\alpha(f)$ and $P_\alpha(f)=\{p\in U \mid \|f-p\|_\alpha=D_\alpha(f)\}$ as functions of α for fixed f. We shall show their continuous dependence on α and differentiability with respect to α.

Key Words: best approximation; peak norm; α-norm; continuity; differentiability.

1. INTRODUCTION

Let $C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$ and U be an n dimensional subspace of $C[0, 1]$. For $0<\alpha\leqslant 1$, the peak norm, or α-norm is defined by

$$\|f\|_\alpha=\frac{1}{\alpha}\sup\left\{\int_A |f|\,d\mu \mid \mu(A)=\alpha,\ A\subset[0,1]\right\},$$

where μ denotes the Lebesgue measure. We say $p_\alpha\in U$ is a best α-norm approximant to f from U if

$$D_\alpha(f)=\|f-p_\alpha\|_\alpha=\inf\{\|f-u\|_\alpha \mid u\in U\}.$$

α-norm and best α-norm approximation were introduced and discussed in [4], and also in [6]. α-norms serve as a bridge between the classical uniform norm and L^1 norm, because it is L^1 norm when $\alpha=1$ and $\lim_{\alpha\to 0}\|f\|_\alpha=\|f\|=\max_{0\leqslant x\leqslant 1}|f(x)|$, the uniform norm of f. Best α-norm approximation has both an L^1-type characterization theorem and alternating

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property [4]. A sufficient condition for the uniqueness of best α-norm approximation is given in [6]: "U is an A-space and $\mu(Z(u)) < \alpha$ for any $0 \not\equiv u \in U$, where $Z(u) = \{x \mid u(x) = 0\}$." This condition becomes that U is an A-space when $\alpha = 1$ and U is a Chebyshev space on (0, 1) when $\alpha \to 0$. Recall that an A-space guarantees the uniqueness of the best L^1 approximation and a Chebyshev space guarantees the uniqueness of the best uniform approximation. Recently the peak L^p norms are studied in [5].

In this paper we shall study $\|f\|_\alpha$, $D_\alpha(f)$ and $P_\alpha(f) = \{p \in U \mid \|f - p\|_\alpha = D_\alpha(f)\}$ as functions of α for a fixed function f. We shall show their continuous dependence on α and differentiability with respect to α.

2. CONTINUITY IN α

We begin with stating some known results:

THEOREM 2.1 [4, 6]. *Let $f \in C[0, 1]$, U, $D_\alpha(f)$ and $P_\alpha(f)$ be defined as above. Then*

(1) *If $0 < \beta < \alpha \leqslant 1$, then*

$$\|f\|_\alpha \leqslant \|f\|_\beta \leqslant \frac{\alpha}{\beta} \|f\|_\alpha$$

and

$$D_\alpha(f) \leqslant D_\beta(f) \leqslant \frac{\alpha}{\beta} D_\alpha(f).$$

(2) *If U is an A-space and $\mu(Z(u)) < \alpha$ for any $0 \not\equiv u \in U$, then there exists a $\delta > 0$ such that the best β-norm approximation of f is unique for all $\beta > \alpha - \delta$, which is denoted by $p_\beta(f)$, and*

$$\lim_{\beta \to \alpha} p_\beta(f) = p_\alpha(f), \qquad 0 < \alpha < 1,$$

and

$$\lim_{\beta \to 1^-} p_\beta(f) = p_1(f).$$

(3) *If U is a Cheyshev space, then*

$$\lim_{\beta \to 0^+} p_\beta(f) = p_0(f),$$

where $p_0(f)$ denotes the unique best uniform norm approximant of f.

We need some more notations.

Let $A_\alpha(f)$ denote any α-norm norming set of f; i.e., $\mu(A_\alpha(f)) = \alpha$ and

$$\frac{1}{\alpha}\int_{A_\alpha(f)} |f| = \|f\|_\alpha.$$

In what follows, we always choose $A_\alpha(f) \subset A_\beta(f)$ whenever $\alpha \leqslant \beta$.

Let

$$h_\alpha(f) = \inf\{h \mid \mu\{x \in [0, 1] \mid |f(x)| \geqslant h\} \leqslant \alpha\}.$$

It is worth noting that for any norming set $A_\alpha(f)$

$$\{x \mid |f(x)| > h_\alpha(f)\} \subset A_\alpha(f) \subset \{x \mid |f(x)| \geqslant h_\alpha(f)\}.$$

Let

$$E(f) = \{x \mid |f(x)| = \|f\|\},$$

where $\|\cdot\|$ denotes the uniform norm.

Also, for simplicity, $\alpha \to 0(1)$ means $\alpha \to 0^+(1^-)$, and $f'_+(x)$ is considered only for $0 \leqslant x < 1$ and $f'_-(x)$ is considered only for $0 < x \leqslant 1$.

THEOREM 2.2. *Let $P_\alpha = P_\alpha(f)$. For $0 \leqslant \alpha \leqslant 1$, we have*

$$\lim_{\beta \to \alpha} \sup_{p \in P_\beta} \inf_{q \in P_\alpha} \{\|p - q\|\} = 0.$$

Proof. Suppose that the above limit does not go to 0, then there exist α_k, $k = 1, 2, ...$ with $|\alpha_k - \alpha| \leqslant \frac{1}{k}$ and $p_k \in P_{\alpha_k}$ such that

$$\inf_{q \in P_\alpha} \{\|q - p_k\|\} > \frac{1}{k}. \tag{1}$$

Since $\{p_k\}$ is bounded, by compactness of a closed bounded set in a finite dimensional space, there is a subsequence $\{p_{k_j}\}$ converging to p. Then, for $0 < \alpha \leqslant 1$,

$$\begin{aligned}\|f - p\|_\alpha &= \lim_{j \to \infty} \|f - p_{k_j}\|_\alpha \leqslant \lim_{j \to \infty} \max\left\{1, \frac{\alpha}{\alpha_{k_j}}\right\} \|f - p_{k_j}\|_{\alpha_{k_j}} \\ &= \lim_{j \to \infty} D_{\alpha_{k_j}}(f) = D_\alpha(f).\end{aligned}$$

The last inequality follows from Theorem 2.1. This means that $p \in P_\alpha$ and it contradicts (1).

For $\alpha = 0$ and $p \notin P_0$,

$$\|f - p\| > \|f - q\| = D_0(f)$$

and there exist $x_0 \in [0, 1]$ and $\varepsilon_0 > 0$ such that

$$|f(x_0) - p(x_0)| > \|f - q\| + 3\varepsilon_0.$$

By the continuity of f and the fact $\lim_{j \to \infty} \|p_{k_j} - p\| = 0$, there exist $m > 0$ such that for $j > m$ and $|x - x_0| < \frac{1}{m}$.

$$|f(x) - p_{k_j}(x)| > \|f - q\| + \varepsilon_0 \qquad \text{and} \qquad \alpha_{k_j} < \frac{1}{m}$$

and then

$$\|f - p_{k_j}\|_{\alpha_{k_j}} \geqslant \|f - q\| + \varepsilon_0 > \|f - q\|_{\alpha_{k_j}}.$$

This contradicts that $p_{k_j} \in P_{\alpha_{k_j}}$.

The next two lemmas show the continuity of $h_\alpha(f - p)$ with $p \in P_\alpha(f)$. These results will be used in proving the differentiability of $D_\alpha(f)$.

LEMMA 2.3. *If $\beta < \alpha$, then*

$$\sup_{q \in P_\alpha(f)} \{h_\alpha(f - q)\} \leqslant \inf_{p \in P_\beta(f)} \{h_\beta(f - p)\}.$$

Proof. For any $p_\alpha \in P_\alpha$ and $p_\beta \in P_\beta$ with $\beta < \alpha$,

$$\begin{aligned}
\int_{A_\alpha(f - p_\alpha)} |f - p_\alpha| = \alpha D_\alpha(f) &\leqslant \int_{A_\alpha(f - p_\beta)} |f - p_\beta| \\
&= \int_{A_\beta(f - p_\beta)} |f - p_\beta| + \int_{A_\alpha(f - p_\beta) - A_\beta(f - p_\beta)} |f - p_\beta| \\
&\leqslant \beta D_\beta(f) + (\alpha - \beta)\, h_\beta(f - p_\beta) \\
&\leqslant \int_{A_\beta(f - p_\alpha)} |f - p_\alpha| + (\alpha - \beta)\, h_\beta(f - p_\beta) \\
&= \int_{A_\beta(f - p_\alpha)} |f - p_\alpha| + (\alpha - \beta)\, h_\alpha(f - p_\alpha) \\
&\quad - (\alpha - \beta)\, h_\alpha(f - p_\alpha) + (\alpha - \beta)\, h_\beta(f - p_\beta)
\end{aligned}$$

$$\leqslant \int_{A_\beta(f-p_\alpha)} |f-p_\alpha| + \int_{A_\alpha(f-p_\alpha)-A_\beta(f-p_\alpha)} |f-p_\alpha|$$

$$-(\alpha-\beta)\, h_\alpha(f-p_\alpha) + (\alpha-\beta)\, h_\beta(f-p_\beta)$$

$$= \int_{A_\alpha(f-p_\alpha)} |f-p_\alpha| + (\alpha-\beta)(h_\beta(f-p_\beta) - h_\alpha(f-p_\alpha)).$$

Thus $h_\beta(f-p_\beta) - h_\alpha(f-p_\alpha) \geqslant 0$.

LEMMA 2.4. *Let* $0 \leqslant \alpha \leqslant 1$ *and* $f \in C[0, 1]$. *Then*

$$\lim_{\beta\to\alpha^+} \sup_{q\in P_\beta(f)} \{|h_\beta(f-q) - \inf_{p\in P_\alpha(f)} \{h_\alpha(f-p)\}|\} = 0 \tag{2}$$

and

$$\lim_{\beta\to\alpha^-} \sup_{q\in P_\beta(f)} \{|h_\beta(f-q) - \sup_{p\in P_\alpha(f)} \{h_\alpha(f-p)\}|\} = 0. \tag{3}$$

Proof. By Lemma 2.3, for $\beta > \alpha$,

$$\sup_{q\in P_\beta(f)} \{|h_\beta(f-q) - \inf_{p\in P_\alpha(f)} \{h_\alpha(f-p)\}|\}$$

$$= \sup_{q\in P_\beta(f)} \{ \inf_{p\in P_\alpha(f)} \{|h_\beta(f-q) - h_\alpha(f-p)|\}\}$$

and for $\beta < \alpha$,

$$\sup_{q\in P_\beta(f)} \{|h_\beta(f-q) - \sup_{p\in P_\alpha(f)} \{h_\alpha(f-p)\}|\}$$

$$= \sup_{q\in P_\beta(f)} \{ \inf_{p\in P_\alpha(f)} \{|h_\beta(f-q) - h_\alpha(f-p)|\}\},$$

and

$$\sup_{q\in P_\beta(f)} \{ \inf_{p\in P_\alpha(f)} \{|h_\beta(f-q) - h_\alpha(f-p)|\}\}$$

$$\leqslant \sup_{q\in P_\beta(f)} \{ \inf_{p\in P_\alpha(f)} \{|h_\beta(f-q) - h_\beta(f-p)|\}\}$$

$$+ \sup_{p\in P_\alpha(f)} \{|h_\beta(f-p) - h_\alpha(f-p)|\}.$$

By Theorem 2.2, the first term of the above expression goes to 0 as $\beta \to \alpha$. Since $P_\alpha(f)$ is a compact set and all functions in this set are continuous, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|p_\alpha(x) - p_\alpha(y)| < \varepsilon$ for any x, $y \in [0, 1]$ with $|x-y| < \delta$ and any $p_\alpha \in P_\alpha(f)$. Thus the second term of the above expression goes to 0 too as $\beta \to \alpha$.

Thus,

$$\lim_{\beta \to \alpha} \sup_{q \in P_\beta(f)} \left\{ \inf_{p \in P_\alpha(f)} \{ |h_\beta(f-q) - h_\alpha(f-p)| \} \right\} = 0.$$

COROLLARY 2.5. *Let* $0 \leqslant \alpha \leqslant 1$ *and* $f \in C[0, 1]$. *If* $h_\alpha(f-p)$ *have the same value for all* $p \in P_\alpha(f)$, *then*

$$\lim_{\beta \to \alpha} \sup_{q \in P_\beta(f)} |h_\beta(f-q) - h_\alpha(f-p)| = 0.$$

When the best α-norm approximation is unique, i.e. $P_\alpha(f)$ is singleton, then $h_\alpha(f-p)$ of course has one value. However, if $P_\alpha(f)$ is not singleton, $h_\alpha(f-p)$ may be different for different $p \in P_\alpha(f)$. See Example 2 in the next section.

3. DIFFERENTIABILITY IN α

THEOREM 3.1. *Let* $f \in C[0, 1]$. *Then, for* $0 < \alpha \leqslant 1$, $\|f\|_\alpha$ *is differentiable with respect to* α, *and*

$$\frac{d}{d\alpha} \|f\|_\alpha = \frac{h_\alpha(f) - \|f\|_\alpha}{\alpha} \leqslant 0.$$

Proof. Choose norming sets $A_\alpha(f)$ and $A_\beta(f)$ such that $A_\alpha(f) \subset A_\beta(f)$ if $\alpha < \beta$, and $A_\beta(f) \subset A_\alpha(f)$ if $\beta < \alpha$. Let $A \Delta B = (A - B) \cup (B - A)$. Then

$$\frac{\|f\|_\beta - \|f\|_\alpha}{\beta - \alpha}$$

$$= \frac{(1/\beta) \int_{A_\beta(f)} |f| - (1/\alpha) \int_{A_\alpha(f)} |f|}{\beta - \alpha}$$

$$= \frac{(1/\beta - 1/\alpha) \int_{A_\alpha(f) \cup A_\beta(f)} |f| + \operatorname{sgn}\{\beta - \alpha\} \min\{1/\beta, 1/\alpha\} \int_{A_\alpha(f) \Delta A_\beta(f)} |f|}{\beta - \alpha}$$

$$= -\frac{1}{\beta\alpha} \int_{A_\alpha(f) \cap A_\beta(f)} |f| + \frac{\min\{1/\beta, 1/\alpha\}}{|\beta - \alpha|}$$

$$\times \left[\int_{A_\alpha(f) \Delta A_\beta(f)} h_\alpha(f) + \int_{A_\alpha(f) \Delta A_\beta(f)} (|f(x)| - h_\alpha(f)) \right].$$

Since $A_\alpha(f) \cap A_\beta(f) = A_\alpha(f)$ when $\alpha \leqslant \beta (= A_\beta(f)$ when $\alpha \geqslant \beta)$, $\mu(A_\alpha(f) \Delta A_\beta(f)) = |\beta - \alpha|$ and $\lim_{\beta \to \alpha} \| \, |f(x)| - h_\alpha(f) \|_{A_\alpha(f) \Delta A_\beta(f)} = 0$, we have

$$\lim_{\beta \to \alpha} \frac{\|f\|_\beta - \|f\|_\alpha}{\beta - \alpha} = -\frac{1}{\alpha^2} \int_{A_\alpha(f)} |f| + \frac{1}{\alpha} h_\alpha(f) = \frac{h_\alpha(f) - \|f\|_\alpha}{\alpha} \leqslant 0.$$

The last inequality follows from the fact $h_\alpha(f) \leqslant \|f\|_\alpha$.

THEOREM 3.2. *Let $f \in C[0, 1]$, then*

$$0 \geqslant \limsup_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant \liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha}$$

$$\geqslant -\left(\inf_{x \in E(f)} \left\{ \limsup_{\substack{h \to 0,\, h > 0 \\ x + h \in [0, 1]}} \left| \frac{f(x+h) - f(x)}{h} \right| (x \neq 1), \right.\right.$$

$$\left.\left. \limsup_{\substack{h \to 0,\, h < 0 \\ x + h \in [0, 1]}} \left| \frac{f(x+h) - f(x)}{h} \right| (x \neq 0) \right\} \right).$$

Proof. Since $\|f\|_\alpha \leqslant \|f\|$, the first two inequalities are obvious. For any $x_0 \in E(f)$, let

$$\limsup_{\substack{h \to 0,\, h > 0 \\ x_0 + h \in [0, 1]}} \left| \frac{f(x_0+h) - f(x_0)}{h} \right| = \lambda.$$

Then for any $\varepsilon > 0$ there exists a δ such that for $0 < h < \delta$,

$$\left| \frac{f(x_0+h) - f(x_0)}{h} \right| < \lambda + \varepsilon,$$

or

$$\frac{|f(x_0+h)| - |f(x_0)|}{h} > -\lambda - \varepsilon.$$

Then, for $\alpha < \delta$,

$$\frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant \frac{1}{\alpha^2} \int_{[x_0, x_0+\alpha]} (|f(x)| - \|f\|) = \frac{1}{\alpha^2} \int_{[x_0, x_0+\alpha]} (|f(x)| - |f(x_0)|)$$

$$\geqslant \frac{1}{\alpha} \int_{[x_0, x_0+\alpha]} \frac{|f(x)| - |f(x_0)|}{x - x_0} > -\lambda - \varepsilon.$$

This shows

$$\liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant - \limsup_{\substack{h \to 0,\, h > 0 \\ x+h \in [0,1]}} \left| \frac{f(x+h) - f(x)}{h} \right|.$$

The proof of

$$\liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant - \limsup_{\substack{h \to 0,\, h < 0 \\ x+h \in [0,1]}} \left| \frac{f(x+h) - f(x)}{h} \right|$$

is similar. Combining these inequalities proves the theorem.

COROLLARY 3.3. *Let* $f \in C[0, 1]$.

(1) *If* $\{x \mid x \in E(f)$ *and either* $f'_+(x)$ *or* $f'_-(x)$ *exists*$\} \neq \phi$, *then*

$$0 \geqslant \limsup_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} \geqslant \liminf_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha}$$

$$\geqslant -\Big(\inf_{x \in E(f)} \{ |f'_-(x)|, |f'_+(x)| \} \Big).$$

(2) *If* $\inf_{x \in E(f)} \{ |f'_-(x)|, |f'_+(x)| \} = 0$, *then*

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = 0.$$

THEOREM 3.4. *Let* $f \in C[0, 1]$. *If both* $f'_+(x)$ *and* $f'_-(x)$ *exist or is* $\pm\infty$ *for any* $x \in E(f)$, *then*

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = \begin{cases} -\dfrac{1}{2 \sum_{x \in E(f)} \left(\dfrac{1}{|f'_+(x)|} + \dfrac{1}{|f'_-(x)|} \right)} & 0 < \min\limits_{x \in E(f)} \{ |f'_+(x)|, |f'_-(x)| \} < \infty, \\ 0 & \inf\limits_{x \in E(f)} \{ |f'_+(x)|, |f'_-(x)| \} = 0, \\ -\infty & \min\limits_{x \in E(f)} \{ |f'_+(x)|, |f'_-(x)| \} = \infty, \end{cases}$$

where $f'_-(0)$ *is not considered if* $0 \in E(f)$ *and* $f'_+(1)$ *is not considered if* $1 \in E(f)$.

Proof. First, if $E(f)$ contains infinite many points, then by compactness, there exists $x_0 \in E(f)$ which is also an accumulation point of $E(f)$. Since both $f'_-(x_0)$ (if $x_0 \neq 0$) and $f'_+(x_0)$ (if $x_0 \neq 1$) exist, at least one of them must be zero. Thus, by Corollary 3.3

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = 0.$$

Now, we assume that $E(f)$ contains only finite points $x_1, x_2, ..., x_k$. For sufficient small $\alpha > 0$, a norming set of f can be expressed as

$$A_\alpha(f) = \bigcup_{i=1}^{k} [s_i, t_i],$$

and $|f(s_{i+1})| = |f(t_i)| = h_\alpha(f)$, $i = 1, ..., k-1$. Also, for sufficient small α,

$$|f(s_1)| = \begin{cases} \|f\| & \text{if} \quad s_1 = x_1 = 0 \\ h_\alpha(f) & \text{if} \quad x_1 > 0 \end{cases}$$

and

$$|f(t_k)| = \begin{cases} \|f\| & \text{if} \quad t_k = x_k = 1 \\ h_\alpha(f) & \text{if} \quad x_k < 1. \end{cases}$$

Then,

$$f'_-(x_i) + o(\alpha) = \frac{f(s_i) - f(x_i)}{s_i - x_i} = \operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{s_i - x_i},$$
$$i = 2, 3, ..., k, \qquad \text{and} \qquad i = 1 \qquad \text{if} \quad x_1 \neq 0$$

and

$$f'_+(x_i) + o(\alpha) = \frac{f(t_i) - f(x_i)}{t_i - x_i} = \operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{t_i - x_i},$$
$$i = 1, 2, ..., k-1, \qquad \text{and} \qquad i = k \qquad \text{if} \quad x_k \neq 1.$$

Solve for $x_i - s_i$ and $t_i - x_i$ from the above equalities and get

$$x_i - s_i = -\operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{f'_-(x_i) + o(\alpha)} = \frac{\|f\| - h_\alpha(f)}{|f'_-(x_i)| + o(\alpha)},$$

$$t_i - x_i = -\operatorname{sgn}(f(x_i)) \frac{h_\alpha(f) - \|f\|}{f'_+(x_i) + o(\alpha)} = \frac{\|f\| - h_\alpha(f)}{|f'_+(x_i)| + o(\alpha)}$$

and

$$\begin{aligned}\alpha &= \sum_{i=1}^{k} [(x_i - s_i) + (t_i - x_i)] \\ &= (\|f\| - h_\alpha(f)) \sum_{i=1}^{k} \left(\frac{1}{|f'_-(x_i)| + o(\alpha)} + \frac{1}{|f'_+(x_i)| + o(\alpha)} \right).\end{aligned}$$

Then,

$$\begin{aligned}\frac{\|f\|_{(\alpha)} - \|f\|}{\alpha} &= \frac{1}{\alpha^2} \int_{A_\alpha(f)} |f(x)| - \|f\| \\ &= \frac{1}{\alpha^2} \sum_{i=1}^{k} \int_{s_i}^{t_i} (x - x_i) \operatorname{sgn}(f(x_i)) \frac{f(x) - f(x_i)}{x - x_i} \\ &= \frac{1}{\alpha^2} \sum_{i=1}^{k} \left[\int_{s_i}^{x_i} (x - x_i) \operatorname{sgn}(f(x_i)) \frac{f(x) - f(x_i)}{x - x_i} \right. \\ &\quad \left. + \int_{x_i}^{t_i} (x - x_i) \operatorname{sgn}(f(x_i)) \frac{f(x) - f(x_i)}{x - x_i} \right] \\ &= \frac{1}{\alpha^2} \sum_{i=1}^{k} \left[\operatorname{sgn}(f(x_i)) \frac{f(x_i^-) - f(x_i)}{x_i^- - x_i} \int_{s_i}^{x_i} (x - x_i) \right. \\ &\quad \left. + \operatorname{sgn}(f(x_i)) \frac{f(x_i^+) - f(x_i)}{x_i^+ - x_i} \int_{x_i}^{t_i} (x - x_i) \right] \\ &= -\frac{1}{2\alpha^2} \sum_{i=1}^{k} \left[\operatorname{sgn}(f(x_i)) \frac{f(x_i^-) - f(x_i)}{x_i^- - x_i} (x_i - s_i)^2 \right. \\ &\quad \left. + \operatorname{sgn}(f(x_i)) \frac{f(x_i^+) - f(x_i)}{x_i^+ - x_i} (t_i - x_i)^2 \right] \\ &= -\frac{1}{2} \sum_{i=1}^{k} \left[(|f'_-(x_i)| + o(\alpha)) \frac{(x_i - s_i)^2}{\alpha^2} \right. \\ &\quad \left. + (|f'_+(x_i)| + o(\alpha)) \frac{(t_i - x_i)^2}{\alpha^2} \right] \\ &= -\frac{1}{2(\sum_{i=1}^{k} (1/(f'_-(x_i) + o(\alpha)) + 1/(f'_+(x_i) + o(\alpha))))^2} \\ &\quad \times \sum_{i=1}^{k} \left[\frac{1}{|f'_-(x_i)| + o(\alpha)} + \frac{1}{|f'_+(x_i)| + o(\alpha)} \right] \\ &= -\frac{1}{2 \sum_{i=1}^{k} (1/(|f'_-(x)| + o(\alpha)) + 1/(|f'_+(x)| + o(\alpha)))},\end{aligned}$$

where $s_i < x_i^- < x_i$ and $x_i < x_i^+ < t_i$ whose existence follows from the Mean Value Theorem for the integral. Thus,

$$\lim_{\alpha \to 0} \frac{\|f\|_\alpha - \|f\|}{\alpha} = -\frac{1}{2\sum_{x \in E(f)} (1/|f'_+(x)| + 1/|f'_-(x)|)}.$$

For the case that $\min_{x \in E(f)} \{|f'_+(x)|, |f'_-(x)|\} = \infty$, the proof is similar.

The following example shows that $\|f\|_{(\alpha)}$ may not be differentiable at $\alpha = 0$ if $f'_+(x)$ or $f'_-(x)$ does not exist for some $x \in E(f)$.

Example 1. Let

$$f(x) = \begin{cases} 0, & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{2^k} + x - \sum_{k=1}^{n} \frac{1}{2^k}, & \sum_{k=1}^{n} \frac{1}{2^k} \leqslant x \leqslant \sum_{k=1}^{n} \frac{1}{2^k} + \frac{1}{2^{n+2}}, \quad n = 1, 2, ... \\ \frac{1}{4} \sum_{k=1}^{n} \frac{1}{2^k}, & \sum_{k=1}^{n} \frac{1}{2^k} + \frac{1}{2^{n+2}} \leqslant x \leqslant \sum_{k=1}^{n+1} \frac{1}{2^k}, \quad n = 1, 2, ... \end{cases}$$

$$\|f\| = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{4} \qquad \text{and} \qquad f'_-(1) \text{ does not exist.}$$

If $\alpha = \sum_{k=n+1}^{\infty} (1/2^k) = 1/2^n$, then

$$\|f\|_\alpha = \frac{1}{1/2^n} \sum_{k=n}^{\infty} \left(\frac{1}{2^{k+1}} \frac{1}{4} \sum_{i=1}^{k-1} \frac{1}{2^i} + \frac{1}{2} \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} \right) \frac{1}{2^{k+2}} \right)$$

$$= 2^n \sum_{k=n}^{\infty} \left(\frac{1}{4} \frac{1}{2^{k+1}} \left(1 - \frac{1}{2^{k-1}} \right) + \frac{3}{8} \frac{1}{4^{k+1}} \right)$$

$$= 2^n \sum_{k=n}^{\infty} \left(\frac{1}{4} \frac{1}{2^{k+1}} - \frac{1}{8} \frac{1}{4^{k+1}} \right) = \frac{1}{4} - \frac{1}{8} \frac{1}{2^n}.$$

If $\alpha=\sum_{k=n+1}^{\infty}(1/2^k)-1/2^{n+2}=1/2^n-1/2^{n+2}$, then

$$\|f\|_\alpha=\frac{1}{1/2^n-1/2^{n+2}}\left[\sum_{k=n}^{\infty}\left(\frac{1}{2^{k+1}}\frac{1}{4}\sum_{i=1}^{k-1}\frac{1}{2^i}+\frac{1}{2}\left(\frac{1}{2^{k+1}}+\frac{1}{2^{k+2}}\right)\frac{1}{2^{k+2}}\right)\right.$$
$$\left.-\frac{1}{2}\frac{1}{2^{n+2}}\left(\frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{2^k}+\frac{1}{4}\sum_{k=1}^{n}\frac{1}{2^k}\right)\right]$$
$$=\frac{2^{n+2}}{3}\left(\sum_{k=n}^{\infty}\left(\frac{1}{4}\frac{1}{2^{k+1}}-\frac{1}{8}\frac{1}{4^{k+1}}\right)-\frac{1}{16}\frac{1}{2^n}+\frac{3}{32}\frac{1}{4^n}\right)$$
$$=\frac{1}{4}-\frac{1}{24}\frac{1}{2^n}.$$

Thus, $\lim_{\alpha\to 0}((\|f\|_\alpha-\|f\|)/\alpha)$ does not exist.

THEOREM 3.5. *Let $f\in C[0,1]$ and $0<\alpha\leqslant 1$. Then*

$$\lim_{\beta\to\alpha^+}\frac{D_\beta(f)-D_\alpha(f)}{\beta-\alpha}=\frac{1}{\alpha}\left(\inf_{p\in P_\alpha(f)}\{h_\alpha(f-p)\}-D_\alpha(f)\right)$$

and

$$\lim_{\beta\to\alpha^-}\frac{D_\beta(f)-D_\alpha(f)}{\beta-\alpha}=\frac{1}{\alpha}\left(\sup_{p\in P_\alpha(f)}\{h_\alpha(f-p)\}-D_\alpha(f)\right).$$

If $h_\alpha(f-p)$ has the same value for all $p\in P_\alpha(f)$, then $D_\alpha(f)$ is differentiable with respect to α and

$$\frac{d}{d\alpha}D_\alpha(f)=\lim_{\beta\to\alpha}\frac{D_\beta(f)-D_\alpha(f)}{\beta-\alpha}=\frac{1}{\alpha}(h_\alpha(f-p)-D_\alpha(f)).$$

Proof. For $\alpha<\beta$, let $\bar{p}_\alpha\in P_\alpha$ such that $h_\alpha(f-\bar{p}_\alpha)=\inf_{p\in P_\alpha(f)}\{h_\alpha(f-p)\}$ and by Lemma 2.4 we can choose $p_\beta\in P_\beta$ such that

$$\lim_{\beta\to\alpha^+}h_\beta(f-p_\beta)=h_\alpha(f-\bar{p}_\alpha). \tag{4}$$

By Theorem 2.2, for each above p_β one can find a $p_{\alpha(\beta)}\in P_\alpha(f)$ such that

$$\lim_{\beta\to\alpha^+}\|p_{\alpha(\beta)}-p_\beta\|=0, \tag{5}$$

and hence

$$\lim_{\beta\to\alpha^+}h_\beta(f-p_{\alpha(\beta)})=h_\alpha(f-\bar{p}_\alpha). \tag{6}$$

Since $P_\alpha(f)$ is a compact set and all functions in this set are continuous, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $p_\alpha \in P_\alpha(f)$ and any x, $y \in [0, 1]$ with $|x - y| < \delta$

$$|p_\alpha(x) - p_\alpha(y)| < \varepsilon,$$

and then by (5)

$$|p_\beta(x) - p_\beta(y)| \leqslant |p_{\alpha(\beta)}(x) - p_{\alpha(\beta)}(y)| + 2 \, \|p_{\alpha(\beta)} - p_\beta\| < \varepsilon$$

for sufficient small δ and $0 \leqslant \beta - \alpha < \delta$.

Thus, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $x \in A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)$ with $|\alpha - \beta| < \delta$

$$\| f(x) - p_\beta(x)| - h_\beta(f - p_\beta)| < \varepsilon.$$

Also by (5)

$$\lim_{\beta \to \alpha^+} |h_\beta(f - p_{\alpha(\beta)}) - h_\beta(f - p_\beta)| = 0. \tag{7}$$

Then

$$\begin{aligned} &\lim_{\beta \to \alpha^+} \sup_{x \in A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} \| f(x) - p_{\alpha(\beta)}(x)| - h_\beta(f - p_{\alpha(\beta)})| \\ &\quad \leqslant \lim_{\beta \to \alpha^+} \sup_{x \in A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} \| f(x) - p_\beta(x)| - h_\beta(f - p_\beta)| \\ &\qquad + \lim_{\beta \to \alpha^+} \|p_{\alpha(\beta)} - p_\beta\| + \lim_{\beta \to \alpha^+} |h_\beta(f - p_{\alpha(\beta)}) - h_\beta(f - p_\beta)| = 0. \end{aligned} \tag{8}$$

Then, by (4), (5), (6), (7), and (8)

$$\begin{aligned} &\int_{A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} |f - p_\beta| \\ &\quad = (\beta - \alpha) \, h_\beta(f - p_\beta) + \int_{A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} (|f - p_\beta| - |f - p_{\alpha(\beta)}|) \\ &\qquad + \int_{A_\beta(f - p_\beta) - A_\alpha(f - p_\beta)} (|f - p_{\alpha(\beta)}| - h_\beta(f - p_{\alpha(\beta)})) \\ &\qquad + (\beta - \alpha)(h_\beta(f - p_{\alpha(\beta)}) - h_\beta(f - p_\beta)) \\ &\quad = (\beta - \alpha) \, h_\beta(f - p_\beta) + o(\beta - \alpha) \end{aligned} \tag{9}$$

and

$$\int_{A_\beta(f-\bar{p}_\alpha)-A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha|$$

$$= (\beta-\alpha)\, h_\alpha(f-\bar{p}_\alpha) + \int_{A_\beta(f-\bar{p}_\alpha)-A_\alpha(f-\bar{p}_\alpha)} (|f-\bar{p}_\alpha| - h_\alpha(f-\bar{p}_\alpha))$$

$$= (\beta-\alpha)\, h_\alpha(f-\bar{p}_\alpha) + o(\beta-\alpha). \tag{10}$$

Then, by (9) and (10)

$$0 \geqslant \beta(\|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta)$$

$$= \int_{A_\beta(f-p_\beta)} |f-p_\beta| - \int_{A_\beta(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha|$$

$$= \int_{A_\alpha(f-p_\beta)} |f-p_\beta| + \int_{A_\beta(f-p_\beta)-A_\alpha(f-p_\beta)} |f-p_\beta|$$

$$- \left[\int_{A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| + \int_{A_\beta(f-\bar{p}_\alpha)-A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| \right]$$

$$= \int_{A_\alpha(f-p_\beta)} |f-p_\beta| + h_\beta(f-p_\beta)(\beta-\alpha) + o(\beta-\alpha)$$

$$- \left[\int_{A_\alpha(f-\bar{p}_\alpha)} |f-\bar{p}_\alpha| + h_\alpha(f-\bar{p}_\alpha)(\beta-\alpha) + o(\beta-\alpha) \right]$$

$$= (h_\beta(f-p_\beta) - h_\alpha(f-\bar{p}_\alpha))(\beta-\alpha) + \alpha\, \|f-p_\beta\|_\alpha - \alpha D_\alpha(f) + o(\beta-\alpha)$$

$$\geqslant (h_\beta(f-p_\beta) - h_\alpha(f-\bar{p}_\alpha))(\beta-\alpha) + o(\beta-\alpha).$$

From the above inequality and (4), we get

$$0 \geqslant \lim_{\beta\to\alpha^+} \frac{\|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta}{\beta-\alpha} \geqslant \lim_{\beta\to\alpha^+} \frac{1}{\beta}(h_\beta(f-p_\beta) - h_\alpha(f-\bar{p}_\alpha)) = 0.$$

This shows

$$\lim_{\beta\to\alpha^+} \frac{\|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta}{\beta-\alpha} = 0. \tag{11}$$

Now,

$$D_\beta(f) - D_\alpha(f) = \|f-p_\beta\|_\beta - \|f-\bar{p}_\alpha\|_\beta + \|f-\bar{p}_\alpha\|_\beta - \|f-\bar{p}_\alpha\|_\alpha \tag{12}$$

and by Theorem 3.1

$$\lim_{\beta \to \alpha^+} \frac{\|f-\bar{p}_\alpha\|_\beta - \|f-\bar{p}_\alpha\|_\alpha}{\beta-\alpha} = \frac{h_\alpha(f-\bar{p}_\alpha) - \|f-\bar{p}_\alpha\|_\alpha}{\alpha}$$

$$= \frac{h_\alpha(f-\bar{p}_\alpha) - D_\alpha(f)}{\alpha}. \tag{13}$$

Finally, combining (11), (12), and (13),

$$\lim_{\beta \to \alpha^+} \frac{D_\beta(f) - D_\alpha(f)}{\beta-\alpha} = \frac{h_\alpha(f-\bar{p}_\alpha) - D_\alpha(f)}{\alpha}$$

$$= \frac{1}{\alpha}\Big(\inf_{p \in P_\alpha(f)} \{h_\alpha(f-p)\} - D_\alpha(f)\Big).$$

Similarly we can prove

$$\lim_{\beta \to \alpha^-} \frac{D_\beta(f) - D_\alpha(f)}{\beta-\alpha} = \frac{1}{\alpha}\Big(\sup_{p \in P_\alpha(f)} \{h_\alpha(f-p)\} - D_\alpha(f)\Big).$$

The following example shows that $h_\alpha(f-p)$ may be different for different $p \in P_\alpha(f)$.

Example 2. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x \leqslant \frac{3}{4} \\ 4-4x, & \frac{3}{4} \leqslant x \leqslant 1, \end{cases} \qquad u(x) = \begin{cases} \frac{1}{4}x - \frac{1}{8}, & 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{2}, & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

and $U = \mathrm{span}\{u\}$.

Then U is a unicity space of α-norm approximation for any α except $\alpha = \frac{3}{4}$ (see [6]). For $\alpha = \frac{3}{4}$, all $cu(x)$, $0 \leqslant c \leqslant 1$ are best α-norm approximation of f from U and $h_\alpha(f-cu)$ ranges from $\frac{3}{4}$ to 1.

Theorem 3.6. *Let $f \in C[0, 1]$.*

(1) *If f and all $u \in U$ satisfy Lipschiz condition, i.e., there exists $L > 0$ such that for any $x, y \in [0, 1]$*

$$|f(x) - f(y)| \leqslant L\,|x-y|$$

and for each $u \in U$, there exists $L_u > 0$ such that for any $x, y \in [0, 1]$

$$|u(x) - u(y)| \leqslant L_u\,|x-y|,$$

then the exists $M > 0$ *such that*

$$|D_\alpha(f) - D_0(f)| \leqslant M\alpha.$$

(2) *If* $P_0(f) = \{p_0\}$ *is singleton,* $u'(x) \in C[0, 1]$ *for all* $u \in U$, *and* $f'(x) - p_0'(x) = 0$ *for all* $x \in E(f - p_0)$, *then*

$$\lim_{\alpha \to 0} \frac{D_\alpha(f) - D_0(f)}{\alpha} = 0.$$

Proof. Since $\bigcup_{0 \leqslant \alpha \leqslant 1} P_\alpha(f)$ is a bounded set in a finite dimensional space and each of them satisfies Lipschiz condition, we can find a number $B > 0$ such that for any $p \in \bigcup_{0 \leqslant \alpha \leqslant 1} P_\alpha(f)$ and $|x - y| \leqslant \alpha$

$$|p(x) - p(y)| \leqslant B\alpha.$$

Let $M = L + B$ and then

$$\begin{aligned} |D_\alpha(f) - D_0(f)| &= D_0(f) - D_\alpha(f) \\ &= \|f - p_0\| - \|f - p_\alpha\| + \|f - p_\alpha\| - \|f - p_\alpha\|_\alpha \\ &\leqslant \|f - p_\alpha\| - \|f - p_\alpha\|_\alpha \leqslant \|f - p_\alpha\| - h(f - p_\alpha) \leqslant M\alpha. \end{aligned}$$

This proves the first part of the theorem. Now, for the second part, by Theorem 2.1 we have

$$0 \leqslant \frac{D_0(f) - D_\alpha(f)}{\alpha}.$$

By Theorem 2.2, we can choose $p_\alpha \in P_\alpha(f)$ for each α so that $\lim_{\alpha \to 0} \|p_\alpha - p_0\| = 0$. Since they are all from a finite dimensional space and have continuous derivatives, $\lim_{\alpha \to \infty} \|p_\alpha' - p_0'\| = 0$. By a property of best uniform approximation there exists $x_0 \in E(f - p_0)$ such that $\|f - p_0\| = |f(x_0) - p_0(x_0)| \leqslant |f(x_0) - p_\alpha(x_0)|$.

Then

$$\begin{aligned} &\frac{\|f - p_0\| - \|f - p_\alpha\|_\alpha}{\alpha} \\ &\quad \leqslant \frac{1}{\alpha}\left[|f(x_0) - p_\alpha(x_0)| - \frac{1}{\alpha} \int_{A_\alpha(f - p_\alpha)} |f - p_\alpha| \right] \\ &\quad \leqslant \frac{1}{\alpha^2} \int_{x_0}^{x_0 + \alpha} (|(f(x_0) - p_\alpha(x_0)) - (f(x) - p_\alpha(x))|) \end{aligned}$$

$$\leqslant \frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(f(x_0)-p_0(x_0))-(f(x)-p_0(x))|)$$

$$+\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|p_\alpha(x_0)-p_0(x_0)|-|p_\alpha(x)-p_0(x)|.$$

For small α, the first term

$$\left|\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(f(x_0)-p_0(x_0))-(f(x)-p_0(x))|)\right|$$

$$\leqslant \max_{x_0\leqslant x\leqslant x_0+\alpha}\left|\frac{(f(x)-p_\alpha(x))-(f(x_0)-p_\alpha(x_0))}{\alpha}\right|.$$

It goes to 0 by the assumption that the derivatives of $f-p_\alpha$ are all zero at any $x\in E(f)$. The second term

$$\frac{1}{\alpha^2}\int_{x_0}^{x_0+\alpha}(|(p_\alpha(x_0)-p_0(x_0))-(p_\alpha(x)-p_0(x))|)$$

$$\leqslant \max_{x_0\leqslant x\leqslant x_0+\alpha}\left|\frac{(p_\alpha(x)-p_0(x))-(p_\alpha(x_0)-p_0(x_0))}{\alpha}\right|$$

also goes to 0 because $\lim_{\alpha\to\infty}\|p'_\alpha-p'_0\|=0$.

The second part of the theorem is proved.

Comparing part (2) of Theorem 3.6 and part (2) of Corollary 3.3, one might ask if the condition $f'(x)-p'_0(x)=0$ for all $x\in E(f-p_0)$ can be replaced by $f'(x)-p'_0(x)=0$ for one $x\in E(f-p_0)$. the following example shows it cannot

EXAMPLE 3. Let

$$f(x)=\begin{cases}1-2x, & 0\leqslant x\leqslant \frac{1}{2}\\ (2x-1)(2x-3), & \frac{1}{2}\leqslant x\leqslant 1,\end{cases}$$

$$u(x)=\begin{cases}1, & 0\leqslant x\leqslant \frac{2}{5}\\ \text{linear}, & \frac{2}{5}\leqslant x\leqslant \frac{3}{5}\\ 3, & \frac{3}{5}\leqslant x\leqslant 1,\end{cases}$$

and $U = \text{span}\{u\}$. Then $p_0(f) = 0$, $D_0(f) = \|f\| = 1$, and $f'(1) - p_0'(f)(1) = 0$, but, for $\alpha \leqslant \frac{1}{4}$,

$$\frac{D_\alpha(f) - D_0(f)}{\alpha} \leqslant \frac{\left\| f + \frac{1}{2}\alpha u \right\|_\alpha - \|f\|}{\alpha}$$

$$= \frac{1}{\alpha}\left(\frac{1}{\alpha}\int_0^\alpha \left(1 - 2x + \frac{1}{2}\alpha\right) dx - 1\right)$$

$$= -\frac{1}{2} < 0.$$

THEOREM 3.7. *Let $f \in C[0, 1]$ and U be a Chebyshev space.*

(1) *If f and all $u \in U$ satisfy Lipschiz condition, then for any $p_\alpha \in P_\alpha(f)$,*

$$\|p_\alpha - p_0\| \leqslant C\alpha, \qquad \textit{for some constant } C.$$

(2) *If $f'(x), u'(x) \in C[0, 1]$ for all $u \in U$ and $f'(x) - p_0'(x) = 0$ for all $x \in E(f - p_0)$, then*

$$\lim_{\alpha \to 0} \frac{\|p_\alpha - p_0\|}{\alpha} = 0.$$

Proof. By the same reason given in the proof of Theorem 3.6, there exists $L > 0$ such that for any $|x - y| \leqslant \alpha$ and any $p \in \bigcup_{0 \leqslant \alpha \leqslant 1} P_\alpha(f)$

$$|f(x) - f(y)| \leqslant \alpha \qquad \text{and} \qquad |p(x) - p(y)| \leqslant \alpha.$$

By the strong uniqueness of the best uniform approximation, there exists $\gamma = \gamma(f) > 0$ such that

$$\|f - u\| \geqslant \|f - p_0\| + \gamma\, \|p_0 - u\|$$

for any $u \in C[0, 1]$. Also, $h(f - p_\alpha) \leqslant \|f - p_\alpha\|_\alpha \leqslant \|f - p_0\|_\alpha \leqslant \|f - p_0\|$.

Now, replace u by p_α and get

$$\|p_0 - p_\alpha\| \leqslant \frac{1}{\gamma}\left(\|f - p_\alpha\| - \|f - p_0\|\right)$$

$$\leqslant \frac{1}{\gamma}\left(\|f - p_\alpha\| - h(f - p_\alpha)\right) \leqslant \frac{2L}{\gamma}\,\alpha.$$

For the second part of the theorem, we have

$$
\begin{aligned}
0 \leqslant \lim_{\alpha\to 0} \frac{\|p_\alpha - p_0\|}{\alpha} &\leqslant \lim_{\alpha\to 0} \frac{1}{\gamma}\left(\frac{\|f-p_\alpha\| - \|f-p_0\|}{\alpha}\right) \\
&\leqslant \frac{1}{\gamma} \lim_{\alpha\to 0} \frac{\|f-p_\alpha\| - \|f-p_\alpha\|_\alpha}{\alpha} \\
&\leqslant \lim_{\alpha\to 0} \frac{1}{\alpha}\int_{x_\alpha}^{x_\alpha+\alpha}\left(\frac{|(f-p_\alpha)(x_\alpha)| - |(f-p_\alpha)(x)|}{\alpha}\right) \\
&\leqslant \max_{x_\alpha\leqslant x\leqslant x_\alpha+\alpha} \frac{|(f-p_\alpha)(x_\alpha)| - |(f-p_\alpha)(x)|}{\alpha} = 0.
\end{aligned}
$$

where $|(f-p_\alpha)(x_\alpha)| = \|f-p_\alpha\|$.

This proves the second part of the theorem.

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